Nowhere Dense Sets

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1 Introduction

The motivation for this essay orginates from two distinct places. The first is the idea of a nowhere dense set, a notion that was briefly introduced in Norms, Metrics and Topologies but not further expanded on. The second being a question I had asked myself in first year, "what function is continuous on the rationals and discontinuous on the irrationals?" After looking into nowhere dense sets in a bit more depth I discovered that they could be used to answer my question, as well as show several other interesting results. The aim of this essay is to present some of these examples. We start by proving the *Baire category theorem*, a very powerful theorem that we will employ as our main tool to understanding first category sets.

2 Baire category theorem

Definition 2.1. Given a metric space (X, d), a subset A is called *dense* in X, if $\overline{A} = X$.

Put otherwise, A is dense in X if and only if $A \cap E \neq \varnothing$ for every non-empty open set E.

Definition 2.2. Given a metric space (X, d), a subset A is called *nowhere dense* in X if $(\overline{A})^{\circ} = \varnothing$.

In other words, A is nowhere dense in X if and only if it is contained in a closed set with empty interior.

It follows immediately from the definition that a subset of a nowhere dense set is nowhere dense. In addition, the closure of a nowhere dense set is nowhere dense. Please refer to Appendix A for some equivalent statements of these definitions that we shall use throughout.

Proposition 2.3. *Let X be a metric space. The finite union of nowhere dense sets remains nowhere dense in X.*

Proof. It is enough to prove the statement for two nowhere dense sets A_1, A_2. It is clear from Lemma A.2 that $U_1 = X \setminus \overline{A_1}$ and $U_2 = X \setminus \overline{A_2}$ are dense and open. The set $U_1 \cap U_2$ is trivially open. We claim it is also dense, so let $E \subset X$ be a non-empty open set. Note that $U_1 \cap E$ is open and non-empty (as U_1 is dense) and as U_2 is also dense, we have that $U_2 \cap (U_1 \cap E)$ is non-empty too. Our choice E was arbitrary and as $(U_2 \cap U_1) \cap E \neq \varnothing$, the claim is true. By De Morgan's Law of Union we have that $U_1 \cap U_2 = X \setminus ((\overline{A_1}) \cup (\overline{A_2}))$ is dense. Appealing to Lemma A.2 again gives that $\overline{A_1} \cup \overline{A_2}$ is nowhere dense. To conclude note that $A_1 \cup A_2 \subset \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ and so $A_1 \cup A_2$ is nowhere dense. $\square$

One may naively assume that the union of countably many nowhere dense remains nowhere dense as well. However, a simple counterexample is to take $X = \mathbb{R}$. The set $\mathbb{Q}$ is a countable union of nowhere dense sets (the rationals are an enumerable union of singletons and singletons are nowhere dense in $\mathbb{R}$). Yet $\mathbb{Q}$ is not nowehere dense $\big((\overline{\mathbb{Q}})^{\circ} = \overline{\mathbb{Q}} = \mathbb{R}\big)$. This example shows the need to define a new class of sets.

Definition 2.4. Given a metric space (X, d), a subset A is said to be,

1. *First category* in X if it can be represented as a countable union of nowhere dense sets.
2. *Second category* in X if it is not of first category.
3. *Residual* in X if it is the complement of a first category set.

It is important to note that it is not immediate from the definition that, a priori, a residual set cannot be also of first category.

Theorem 2.5. *Let X be a metric space. Then,*

1. *Any subset of a set of first category remains of first category.*
2. *The union of countably many first category sets is also of first category.*

Proof.
(1): Let A be a set of first category. We have a countable collection of nowhere dense sets denoted by E_n such that $A = \bigcup_{n\in\mathbb{N}} E_n$. Let $B \subset A$, we have:

$$B = A \cap B = \left(\bigcup_{n\in\mathbb{N}} E_n\right) \cap B = \bigcup_{n\in\mathbb{N}} (E_n \cap B).$$

It follows that B can written as the countable union of nowhere dense sets since $(E_n \cap B) \subset E_n$ and a subset of a nowhere dense set is nowhere dense.

(2): Let $\{A_n : n \in \mathbb{N}\}$ be countable collection of first category sets. So we have that each $A_n = \bigcup_{j\in J_n} E_{n,j}$, where the $E_{n,j}{}'s$ are all nowhere dense and J_n is a countable indexing set for all $n \in \mathbb{N}$. This yields:

$$A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{j\in J_n} E_{n,j}\right).$$

Since the countable union of countable sets is countable we have shown that A is of first category. □

We now introduce the main result that we will use throughout the remainder of the essay. It was first proven for $\mathbb{R}^n$ in 1899 by René-Louis Baire. The more general statement for metric spaces was first proven in 1914 by Felix Hausdorff.

Theorem 2.6 (Baire Category Theorem)**.** *Baire (1899) If X is a complete metric space we have the following:*

1. *Every residual set in X is dense in X.*
2. *For every countable family of dense open sets $\{G_n\}$ we have $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X.*
3. *For every countable family of closed sets $\{F_n\}$ such that $X = \bigcup_{n\in\mathbb{N}} F_n$ we have $\bigcup_{n\in\mathbb{N}} (F_n)^{\mathrm{o}}$ is dense in X.*

Remark. Statement 1 of the theorem is equivalent to first category sets having empty interior.

Proof. Let A be a first category subset of X and by Lemma A.1 We have:

$$\overline{X \setminus A} = X \iff X \setminus \overline{X \setminus A} = \varnothing \iff A^{\mathrm{o}} = \varnothing.$$

□

Remark. Statement 3 of the theorem tells us that tells us that a complete metric space is necessarily second category in itself.

Proof. This follows by noting that statement 3 can be weakened to saying that there exists at least one F_j which has nonempty interior. □

This also tells us that in complete metric spaces residual sets must be second category as well as dense since otherwise we would contradict statement 3.

To prove the Baire category theorem we will require a Lemma and the following Theorem, first proven by Georg Cantor.

Lemma 2.7. *The boundary of a closed set is nowhere dense.*

Proof. Let A be closed and recall that the boundary is closed. It is enough to show that $(\partial A)^{\mathrm{o}} = \varnothing$. Let $U \subset \partial A$ be open. Note that $\partial A \subset \overline{A} = A$ implying that $U \subset A$. We also know that $U \subset A^{\mathrm{o}}$ since U is open. We must have then that $U \subset \partial A \cap A^{\mathrm{o}} \implies U \subset \overline{A} \setminus A^{\mathrm{o}} \cap A^{\mathrm{o}} = \varnothing$. Hence $U = \varnothing$. □

Definition 2.8. If A is a non empty subset of some metric space (X,d) then the diameter of A is given by

$$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}.$$

Theorem 2.9 (Cantor's Intersection Theorem)**.** *Let X be a complete metric space and let (F_n) a decreasing sequence of nonempty closed subsets of X with $\operatorname{diam}(F_n) \to 0$. Then, there exists a point $x \in X$ such that*

$$\bigcap_{n\in\mathbb{N}} F_n = \{x\}, \text{ thus } \bigcap_{n\in\mathbb{N}} F_n \neq \varnothing.$$

Proof. (Sokal 2012, p. 4) For each $n \in \mathbb{N}$ choose $x_n \in F_n$. Since (F_n) is decreasing, for all $i \geq n$ we have $x_i \in F_n$. The sequence (x_n) is Cauchy since if $m,n \geq N$ we have $d(x_m,x_n) \leq \operatorname{diam}(F_n)$ which tends to zero as $N \to \infty$. As X is complete the sequence (x_n) has a limit which we will call x. Note that $x_n \in F_n$ for all $n \geq N$ and as F_n is closed we see $x \in F_n$. As this is true for all N we have that $x \in \bigcap_{N\in\mathbb{N}} F_n$. We can conclude by noting that $\operatorname{diam}(\bigcap_{N\in\mathbb{N}} F_n) \leq \inf_{N\geq 1} \operatorname{diam}(F_N) = 0$, which implies that $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$ as required. ☐

With this theorem we are well poised to complete the proof of the *Baire Category Theorem.*

Proof. The proof follows the one given in (Giles 2000, p. 142) with alterations made to fill the gaps left by the author in showing the statements are equivalent. The large unions and intersection symbols in this proof should be taken to be countable. Ideally we would like to only have to tackle one of the three results. So lets first show that they are all equivalent.

(1) $\Rightarrow$ (3): F_n is closed so by Lemma 2.7 we have that ∂F_n is nowhere dense, hence $\bigcup \partial F_n$ is of first category in X. Using our assumption that every residual set is dense we have that $X \setminus \bigcup \partial F_n$ is dense in X. Again using the fact that F_n is closed we have that $\partial F_n = F_n \setminus (F_n)^{\mathrm{o}}$. We now claim $X \setminus \bigcup \partial F_n \subset \bigcup {F_n}^{\mathrm{o}}$. To show this take some $x \in X \setminus \bigcup \partial F_n$, since $X = \bigcup F_n$ we know there must exist some $n \in \mathbb{N}$ such that $x \in F_n$. See that this $x \notin \partial F_n$ for all $n \in \mathbb{N}$. Hence $x \in F_n \setminus \partial F_n = {F_n}^{\mathrm{o}} \subset \bigcup {F_n}^{\mathrm{o}}$. Our choice of x was arbitrary so we must have $X \setminus \bigcup \partial F_n \subset \bigcup {F_n}^{\mathrm{o}}$. It follows immediately that $\bigcup {F_n}^{\mathrm{o}}$ is dense in X.

(3) $\Rightarrow$ (2): We can express X as follows $X = (\overline{\bigcap G_n}) \cup (\bigcup (X \setminus G_n))$, to see why note that the RHS will contain every x that is not in G_n for all $n \in \mathbb{N}$ and these missing x clearly lie in LHS. We have X to be the union of countably many closed sets. By (3) we have $(\overline{\bigcap G_n})^{\mathrm{o}} \cup (\bigcup (X \setminus G_n)^{\mathrm{o}})$ to be dense in X. However since G_n is dense in X by Lemma A.4 we have that $(X \setminus G_n)^{\mathrm{o}} = \varnothing$. Hence $(\overline{\bigcap G_n})^{\mathrm{o}}$ must be dense in X. Which implies that $\bigcap G_n$ is dense in X. To see this we prove the contrapositive, $\bigcap G_n$ not dense in X $\Rightarrow (\overline{\bigcap G_n})^{\mathrm{o}}$ not dense in X. By the assumption we have $\overline{\bigcap G_n} \neq X$. Since $(\overline{\bigcap G_n})^{\mathrm{o}} \subset \overline{\bigcap G_n}$ we see that $\overline{(\overline{\bigcap G_n})^{\mathrm{o}}} \subset \overline{\overline{\bigcap G_n}} = \overline{\bigcap G_n} \neq X$. Hence $(\overline{\bigcap G_n})^{\mathrm{o}}$ is not dense in X.

(2) $\Rightarrow$ (1): We begin by considering countable $\{E_n\}$ nowhere dense sets in X, by definition $\bigcup E_n$ is of first category in X. By Lemma A.2 we have that $X \setminus \overline{E_n}$ is both open and dense in X and therefore by the assumption we have that $\bigcap (X \setminus \overline{E_n})$ to be dense in X. Showing that $\bigcap (X \setminus \overline{E_n}) \subset (X \setminus \bigcup E_n)$ would conclude the proof since we would then have that $(X \setminus \bigcup E_n)$ is dense in X. To prove the claim take some arbitrary $x \in \bigcap (X \setminus \overline{E_n})$. We have that $x \in (X \setminus \overline{E_n})$ for all $n \in \mathbb{N}$, equivalently, for all $n \in \mathbb{N}$ $x \notin \overline{E_n}$. Since $E_n \subset \overline{E_n}$ it follows that for all $n \in \mathbb{N}$ $x \notin E_n$. But then $x \notin \bigcup E_n$ and so $x \in (X \setminus \bigcup E_n)$ with the result following since x was chosen arbitrarily.

We now directly prove that in a complete metric space we have (3)
Consider a countable family of dense open sets $\{G_n\}$ in X. We will show that for some arbitrary $x \in X$ and $r > 0$, we have $(\bigcap G_n) \cap \mathbb{B}(x,r) \neq \varnothing$.
Each G_i is dense in X, so starting with $i = 1$ we have that there exists $x_1 \in G_1 \cap \mathbb{B}(x,r)$ such that if we choose $0 < r_1 < \frac{r}{2}$ then we have

$$\overline{\mathbb{B}(x_1,r_1)} \subset G_1 \cap \mathbb{B}(x,r).$$

We see that since G_2 is dense in X there exists $x_2 \in G_2 \cap \mathbb{B}(x_1, r_1)$ such that if we now choose $0 < r_2 < \frac{r}{4}$ we have that

$$\overline{\mathbb{B}(x_2, r_2)} \subset G_1 \cap G_2 \cap \mathbb{B}(x, r) \cap \mathbb{B}(x_1, r_1).$$

Now proceeding inductively we have that there exists $x_n \in G_n \cap \mathbb{B}(x_{n-1}, r_{n-1})$ such that for $0 < r_n < \frac{r}{2^n}$ we have that

$$\overline{\mathbb{B}(x_n, r_n)} \subset G_1 \cap G_2 \cap \ldots \cap G_n \cap \mathbb{B}(x, r) \cap \mathbb{B}(x_{n-1}, r_{n-1}).$$

Note that we now have a decreasing sequence of closed subsets whose diameters tend to zero, namely $\left(\overline{\mathbb{B}(x_n, r_n)}\right)_0^\infty$ which, alongside the fact we are working in a complete metric space, allows us to utilise *Cantor's intersection theorem* .That is, there exists some $y \in X$ such that $y \in \bigcap \overline{\mathbb{B}(x_n, r_n)}$. But then we have that $y \in \bigcap G_n \cap \mathbb{B}(x, r)$ thus showing it is non empty as required.

$\square$

3 Transcendence

The Baire category theorem is an existence theorem. That is, if we can show that the set of numbers in a given interval which do not have a certain property is a set of first category. Then we know that there must exist points of that interval that do have the wanted property and in fact,"most" points of the interval have the property.
To this end we begin by showing the existence of transcendental numbers and the fact that almost all numbers are transcendental

Definition 3.1. A number $\alpha \in \mathbb{C}$ is called *algebraic* if it satisfies an equation of the form

$$k_0 + k_1\alpha + k_2\alpha^2 + ... + k_n\alpha^n = 0$$

with integer coefficients, not all zero. Any number that that is not algebraic is called transcendental.

Definition 3.2. The degree of an algebraic number α is the smallest n $\in \mathbb{Z}^+$ such that α satisfies an equation of degree n.

For example any rational number p/q is algebraic with degree 1 $(qx - p)$, $\sqrt{3}$ is algebraic with degree 2 $(x^2 - 3)$, and $\sqrt{5} + \sqrt{6}$ is algebraic with degree 4 $(x^4 - 22x^2 + 1)$.

The existence of transcendental numbers is due to Joseph Liouville who in 1844 first gave a class, *très étendue*, as described in the title of his paper which didn't satisfy the above definiton. His proof stems from the following

Lemma 3.3. *(Liouville 1844) For any algebraic number α of degree $n > 1$ there exists $c \in \mathbb{Z}^+$(dependant on α) such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{cq^n}$$

for all rational p/q, $q > 0$

Proof. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n for which $f(\alpha) = 0$. Let c be a positive integer such that $\left|f'(x)\right| \leq c$ whenever $|\alpha - x| \leq 1$ Then by the *Mean Value Theorem* we have,

$$\left|f(\alpha) - f(x)\right| = \left|f'(d)\right| |\alpha - x| \,,$$

for some d between α and x. Therefore we can apply our bound[1] for f' in conjuction with $f(\alpha) = 0$ to yield

$$\left|f(x)\right| \leq c\,|\alpha - x| \,. \tag{1}$$

We want to show $\left|\alpha - p/q\right| > 1/cq^n$ This is clearly true if $\left|\alpha - p/q\right| > 1$. Hence we can assume that $\left|\alpha - p/q\right| \leq 1$. So by (1) we have that $\left|f(p/q)\right| \leq c\left|\alpha - p/q\right|$ and therefore with $q > 0$ we see

$$\left|q^n f(p/q)\right| \leq cq^n \left|\alpha - p/q\right| . \tag{2}$$

Since $q^n f(p/q)$ is an integer and $f(p/q) \neq 0$ as otherwise α would satisfy an equation with degree less than n. We have that the LHS of (2) is at least 1 and we can conclude the proof by noting equality cannot hold as α is irrational.

□

We now have the tools to construct our first transcendental number, first proven by Liouville in 1851!

Proposition 3.4. *(Liouville 1851)*

$$L = \sum_{m=0}^{\infty} \frac{1}{10^{m!}} \text{ is transcendental.}$$

Proof. Suppose L is an algebraic number. Clearly L cannot be rational since it's decimal expansion is neither finite nor recurs, hence L has degree $n > 1$. For $k \in \mathbb{Z}^+$ define the integers p_k, q_k as:

$$p_k = 10^{k!}\Big(\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{k!}}\Big) \quad \text{and} \quad q_k = 10^{k!}.$$

Now

$$\begin{aligned}
\left|L - \frac{p_k}{q_k}\right| &= \sum_{m=k+1}^{\infty} \frac{1}{10^{m!}} \\
&= \frac{1}{10^{(k+1)!}}\Big(1 + \frac{1}{10^{k+2}} + \frac{1}{10^{k+2}10^{k+3}} + \frac{1}{10^{k+2}10^{k+3}10^{k+4}} + \cdots\Big) \\
&< \frac{1}{10^{(k+1)!}}\Big(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\Big) \\
&= \frac{10/9}{10^{(k+1)!}} \\
&< \frac{2}{10^{(k+1)!}}.
\end{aligned}$$

By Lemma 3.1, there exists c such that for all k,

$$\frac{2}{10^{(k+1)!}} \geq \frac{1}{c(10^{k!})^n}$$

and consequently

$$2c \geq 10^{k!(k+1-n)}$$

which for large enough k gives us our contradiction since c is a fixed finite value, hence L is transcendental. □

Numbers such as the one above are called Liouville numbers.

[1]Note that c was chosen to be an integer for simplicity, a stronger bound can in fact be achieved explicitly without too much difficulty, see (Baker 1975, p. 1-2)

Definition 3.5. A number α is a Louville number if α is irrational and for each $n \in \mathbb{Z}^+$ there exists integers p and q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n} \quad \textit{and} \quad q > 1.$$

Theorem 3.6. *Every Liouville number is transcendental*

Proof. (Oxtoby 1980, p. 7) Suppose there exists some Liouville number α that is algebraic and of degree n. Clearly $n > 1$ as α is irrational. Therefore we can apply Lemma 3.1 and say there is some $c > 0$ such that

$$\left|\alpha - p/q\right| > 1/cq^n \tag{3}$$

for all $p, q \in \mathbb{Z}$ and with $q > 0$. Now choose some $k \in \mathbb{Z}^+$ such that

$$2^k \geq c2^n. \tag{4}$$

Since α is Liouville, there exists $p, q \in \mathbb{Z}$ with $q > 1$ such that

$$\left|\alpha - p/q\right| < 1/q^k. \tag{5}$$

Combining (3) and (5) we get that $1/q^k > 1/cq^n$ implying that $c > q^{k-n}$ which when combined with (4) and the fact that q is an integer greater than 1 gives the contradiction,

$$c > q^{k-n} \geq 2^{k-n} \geq c.$$

□

But exactly how common are Liouville numbers? Using the theory we have developed thus far it is natural to look at the set of numbers which are not Liouville.

Theorem 3.7. *The set L of Liouville numbers is a residual set in $\mathbb{R}$.*

Proof. We expand slightly on the proof given in (Oxtoby 1980, p. 8), by Definition 3.5 we have that

$$L = \Big(\mathbb{R} \setminus \mathbb{Q}\Big) \cap \bigcap_{n \in \mathbb{N}} U_n. \tag{6}$$

Where U_n is defined as:

$$U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \Big\{x \in \mathbb{R} : \left|x - \frac{p}{q}\right| < \frac{1}{q^n}\Big\} = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \Big(-\frac{1}{q^n} + \frac{p}{q}, \frac{1}{q^n} + \frac{p}{q}\Big).$$

U_n is clearly open as it's a union of open sets. Additionally, we have that U_n contains every element of the form p/q , $q \geq 2$, therefore $\mathbb{Q} \subset U_n$. It now follows that,

$$\mathbb{Q} \subset U_n \subset \overline{U_n} \implies \overline{\mathbb{Q}} \subset \overline{\overline{U_n}} = \overline{U_n} \implies \mathbb{R} \subset \overline{U_n}.$$

As $\mathbb{R}$ is the entire space we have $\overline{U_n} = \mathbb{R}$, hence U_n is a dense open set and so by Lemma A.3 it's complement is nowhere dense. Now see that the complement of (6) is:

$$\mathbb{R} \setminus L = \mathbb{Q} \cup \bigcup_{n \in \mathbb{N}} \Big(\mathbb{R} \setminus U_n\Big).$$

Hence the second statement of theorem 2.5 tells us that we have $\mathbb{R} \setminus L$ is of first category and so by the Baire category theorem we have that L is dense in $\mathbb{R}$.

□

We have found that in any interval we take, we will always find Liouville numbers and thus transcendental numbers. They are the "general case" in the sense of category. One might wish that showing numbers are liouville numbers might give a good algorithm to finding transcendentals, alas, there are uncountably many transcendentals that are not Liouville. A liouville number is a number that allows for the ultimate accuracy when approximating by rational numbers(the inequality holds for all positive integers n), but this isn't the case for most. The infamous e for example is transcendental[3] but not liouville[4].

This indicates that whilst the set of Liouville numbers is "large" from the point of view of category, under a different lense the story might be different. We will explore this idea further in Chapter 5. However, before that, we give 2 example applications in real analysis. The first of which is another existence argument, the second gives another look at the idea of size from a category point of view.

4 Functions on $\mathbb{R}$

4.1 Proving a function does not exist

We begin this section by introducing a new notion of continuity which will prove useful in a short while.

Definition 4.1. Given a real function f on $\mathbb{R}$, for any bounded interval J we define $\omega(f, J)$ to be the oscillation of f over J as

$$\omega(f, J) \equiv \sup\{\big|f(x) - f(y)\big| : x, y \in J\},$$

and for $x_0 \in \mathbb{R}$ we define $\omega(f, x_0)$, the oscillation of f at x_0 as

$$\omega(f, x_0) \equiv \inf\{\omega(f, I) : \text{all such I containing } \mathrm{x_0}\}.$$

Theorem 4.2. *f is continuous at x_0 if and only if $\omega(f, x_0) = 0$.*

Proof. $\Longrightarrow$: Suppose f is continuous at x_0. Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow \big|f(x) - f(x_0)\big| < \frac{\varepsilon}{2}.$$

Suppose $x, y \in (x_0 - \delta, x_0 + \delta)$ then we have $|x - x_0| < \delta$ and $|y - x_0| < \delta$ and now using

$$\big|f(x) - f(y)\big| \leq \big|f(x) - f(x_0)\big| + \big|f(y) - f(x_0)\big| < \varepsilon,$$

it follows that $\omega(f, x_0) < \varepsilon \Rightarrow \omega(f, x_0) = 0$.

$\Longleftarrow$: Conversely suppose $\omega(f, x_0) = 0$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\omega(f, (x_0 - \delta, x_0 + \delta)) < \varepsilon$ so then

$$|x - x_0| < \delta \Rightarrow \big|f(x) - f(x_0)\big| < \varepsilon,$$

hence showing that f is continuous at x_0. $\square$

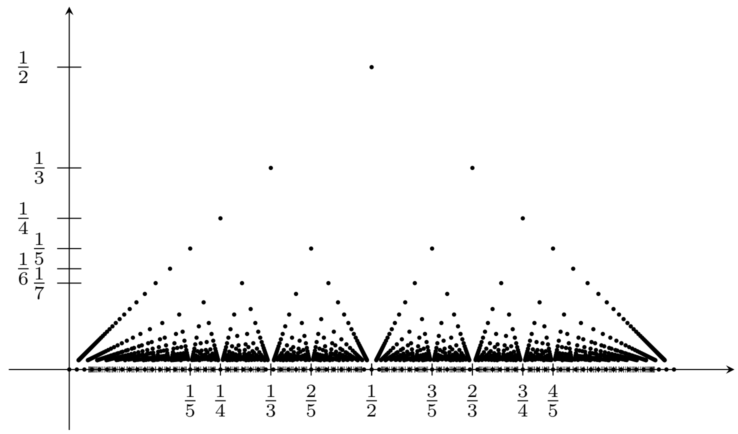
Note that this definition of continuity gives us a quantative estimate on the size of the discontinuity.

We now shift our attention to Thomae's function defined on $\mathbb{R}$ given by

[3] A proof of this is given in full in Appendix B

[4] A proof of this can be found in (Baker 1975, p.86, p.103)

$$f(x) = \begin{cases} \frac{1}{q} & \text{for rational } x = \frac{p}{q}, \text{with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime} \\ 0 & x \notin \mathbb{Q} \end{cases}$$



This function, typically introduced in a first course in real analysis exhibits the pointwise nature of continuity. It is a common exercise to show that the function is continuous at every irrational and discontinuous at every rational. A natural question to ask after proving this is if there exists a function that has the opposite property. That is, can we find a function continuous on the rationals and discontinuous on the irrationals? After trying to construct one for a while one may start to suspect that such a function does not exist.

Theorem 4.3. *For a real function on $\mathbb{R}$, continuous at the points of a dense set C, the set D of points of discontinuity is of first category.*

Proof. (Giles 2000, p.144) Since f is continuous at x_0 if and only if $\omega(f, x_0) = 0$ let us define for all $n \in \mathbb{N}$

$$E_n \equiv \{x \in \mathbb{R} : \omega(f, x) \geq \frac{1}{n}\}.$$

Let us first show E_n is closed, consider a limit point l of E_n. For any bounded interval I such that $l \in I$ there exists an $x \in E_n$ yielding

$$\omega(f, I) \geq \omega(f, x) \geq \frac{1}{n}.$$

However $\omega(f, l)$ is just the infimum of $\omega(f, I)$ and so we have $\omega(f, l) \geq \frac{1}{n}$. So $l \in E_n$ and thus E_n is closed. Clearly $D = \bigcup(E_n : n \in \mathbb{N})$ and now we see our end goal, if we can show each E_n has empty interior we would immediately have D to be of first category since E_n is closed. So let us suppose for a contradiction that there exists $n \in \mathbb{N}$ such that ${E_n}^{\circ} \neq \varnothing$. Then there exists some open subset $u \subset E_n$, but then we must have $u \subset D$ and so $D^{\circ} \neq \varnothing$. Now by Lemma A.1 we see that $D^{\circ} = X \setminus \overline{X \setminus D} = X \setminus \overline{C} \neq \varnothing$. Which contradicts the fact C is dense since we have $\overline{C} \neq X$. So we have that for all $n \in \mathbb{N}$, ${E_n}^{\circ} = \varnothing$. Since the E_n are closed they must all be nowhere dense, hence D is of first category.

□

This Theorem immediately answers our question, it is not possible to construct a real function on $\mathbb{R}$ that is continuous on the rationals and discontinuous at the irrationals. This follows since the set $\mathbb{Q}$, whilst dense, is of first category in $\mathbb{R}$. Then by the *Baire Category Theorem* that the residual set $\mathbb{R} \setminus \mathbb{Q}$ is of second category.

4.2 Nowhere Differentiable Functions

In elementary calculus most of the continuous functions one encounters are well behaved in the sense that they are differentiable everywhere. A few examples crop up such as $|x|$ where we have a singular point of non-differentiability and this can lead to the intuition that continuous functions do not have more than a finite number of points where they are not differentiable or in the absolute worst case a countable set of points. This was the prevailing opinion of most mathematicians in the early nineteenth century, with the famous physicist and mathematician André-Marie Ampère giving an attempted proof in his 1806 paper. So it came as quite a suprise when in 1872 Karl Weierstrass presented a function which was continuous everywhere but differentiable nowhere. A historical quirk is that unlike often stated, Weierstrass was not the first to find such a function, rather the first to officially publish one. Several other earlier examples exist, the earliest believed to be have been constructed by Bernard Bolzano around 1830, which was finally published posthumously in 1922.

Theorem 4.4. *Let $0 < a < 1$ be a real number and let b be a positive odd integer. If $ab > 1$ and $\frac{2}{3} > \frac{\pi}{ab-1}$, then the Weierstrass funcion:*

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

is continuous on $\mathbb{R}$ but nowhere differentiable on $\mathbb{R}$.

The proof whilst interesting is too long to include here and so refer the reader to (Pedersen 2015, p.238-240). We note that in the proof it is also shown that $W(x)$ has unbounded difference quotients on $\mathbb{R}$, a fact we shall soon exploit.

It can be even more suprising that from the point of view of category, almost all continuous functions are nowhere differentiable. This is encapsulated by the following Theorem proven by Stefan Banach in 1931.

Theorem 4.5. *Let $A = \{f \in C([0,1]) : \text{there exists an } x_0 \in [0,1] \text{ such that } f'(x_0) \text{ exists}\}$. A is a set of first category.*

Proof. The proof follows the one given in (Giles 2000, p.147-148) but with the Weierstrass function rather than the saw-tooth function. This choice was motivated by (Vesneske 2019, p.15).

The general strategy for this proof is very similar to previous ones. We wish to define a set that is a countable union of nowhere dense sets. With this in mind it is natural to define the sets for $N \in \mathbb{N}$ as

$$E_N = \{f \in C([0,1]) : \textit{there exists an } x_0 \in [0,1] \textit{ such that } \left|f(x) - f(x_0)\right| \leq N\left|x - x_0\right| \textit{for all } x \in [0,1]\}$$

Our first step is to check this is a useful way to construct our sets, we want for $A \subset \bigcup_{N\in\mathbb{N}} E_N$ since then if we can show the right is a first category set then the result follows by statement 1 of Theorem 2.5.

Let $f \in A$ and fix x_0 such that $f'(x_0)$ exists. Then for some $\delta > 0$ such that $0 < |x - x_0| < \delta$ we have that

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| < \left|f'(x_0)\right| + 1.$$

What about when $|x - x_0| \geq \delta$? Since f is uniformly continuous there exists a bound for f, let us denote this with M, such that $\left|f(x)\right| \leq M$ for all $x \in [0,1]$. So we have

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq \frac{\left|f(x)\right| + \left|f(x_0)\right|}{|x - x_0|} \leq \frac{2M}{\delta}.$$

We now let $N \geq \max\{\left|f'(x_0)\right| + 1, \frac{2M}{\delta}\}$, so that for all $x \in [0,1]$ we have that $\left|f(x) - f(x_0)\right| \leq N\left|x - x_0\right|$. Hence $f \in \bigcup_{N\in\mathbb{N}} E_N$ and since f was chosen arbitrarily we are done.

To show that A is indeed a subset of a first category set requires us to check that for each $N \in \mathbb{N}$ E_N is both closed and has empty interior. We will require an additional Lemma.

Lemma 4.6. *Let $f_n : [a,b] \mapsto \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f : [a,b] \mapsto \mathbb{R}$ and x_n (where $x_n \in [a,b]$) converges to x, then $f_n(x_n)$ converges to $f(x)$.*

Proof. Note that we have the following inequality

$$\begin{aligned}\left|f_n(x_n) - f(x)\right| &\le \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x)\right| \\ &\le \|f_n - f\|_\infty + \left|f(x_n) - f(x)\right|.\end{aligned}$$

Let $\varepsilon > 0$. As $f_n \rightrightarrows f$ there exists N such that $\|f_n - f\|_\infty \le \frac{\varepsilon}{2}$ for all $n > N$. We see from a standard result in analysis that f must be continuous since a sequence of continuous functions converges to it. So given ε there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $\left|f(x_n) - f(x)\right| < \frac{\varepsilon}{2}$. Since $x_n \longrightarrow x$ there exists K such that $|x - x_0| < \delta$ for all $n > K$. So if we take $n > \max\{N, K\}$ then we must have that $\left|f_n(x_n) - f(x)\right| < \varepsilon$. □

With this in mind we proceed with showing that the E_N are closed. Consider a limit point f of one of the E_N. Then we have a sequence $\{f_n\}$ in E_N such that $\{f_n\}$ converges to f. For each n let us choose $x_n \in [0,1]$ to be the x_0 corresponding to each f_n in the definition of E_n. Thus for each n we have that $\left|f_n(x) - f_n(x_n)\right| \le n\,|x - x_n|$ holds true for all $x \in [0,1]$. Since each $x_n \in [0,1]$ the sequence is trivially bounded, this allows us to apply the *Bolzano-Weierstrass Theorem* to $\{x_n\}$. We now have a convergent subsequence of $\{x_n\}$, $\{x_{nk}\}$, that converges to a point $x_0 \in [0,1]$. For $k \in \mathbb{N}$, let $y_k := x_{nk}$ and $g_k := f_{nk}$. We have that $\{f_{nk}\}$ converges to f and $\{y_k\}$ converges to x_0. We can now apply Lemma 4.6 to find that $g_k(y_k)$ converges to $f(x_0)$.

Suppose $x \in [0,1]$ but $x \ne x_0$. Then there must exist $K \in \mathbb{N}$ such that $y_k \ne x$ for all $k > K$. Hence for all $x \in [0,1] \setminus \{x_0\}$ we have that

$$\left|\frac{g_k(y_k) - g_k(x)}{y_k - x}\right| \le N \Longrightarrow \left|\frac{f(x_0) - f(x)}{x_0 - x}\right| \le N.$$

for large enough k. So we see that for all $x \in [0,1]$ it holds true that $\left|f(x) - f(x_0)\right| \le N\,|x - x_0|$. Showing that $f \in E_N$, and since f was chosen arbitrarily we have that the E_N contain all their limit points and are thus closed.

Our final step is to show that E_N has an empty interior for each $N \in \mathbb{N}$. To do this we will show that for any epsilon neighbourhood of an arbitrary $g \in C([0,1])$ there exists a $\psi \in C([0,1])$ such that $\psi \notin E_N$. Our aim then is to show $d_\infty(g, \psi) < \varepsilon$. We first employ the *Weierstrass Approximation Theorem* , this tells us that for $\varepsilon > 0$ there exists some polynomial p such that $\left|p(x) - g(x)\right| < \frac{\varepsilon}{2}$ for all $x \in [0,1]$. We now recall that the Weierstrass function $W(x)$ has unbounded difference quotients for all x. But note that multiplying $W(x)$ by any constant c will yield the function $cW(x)$ which will also have unbounded difference quotients for every x. With this in mind let us define $M = \sup_{x\in[0,1]}\{W(x)\}$ and let $\psi(x) = p(x) + \frac{\varepsilon}{2M}W(x)$. It's easy to see that $\psi \in C([0,1])$ since both p and W are. Also note that ψ will have unbounded difference quotients on $[0,1]$ by construction. Therefore $\psi \notin E_N$ for all $N \in \mathbb{N}$. To conclude we see that

$$\begin{aligned}d_\infty(g, \psi) &= \sup_{x\in[0,1]}\left\{\left|g(x) - \psi(x)\right|\right\} \\ &= \sup_{x\in[0,1]}\left\{\left|g(x) - p(x) - \frac{\varepsilon}{2M}W(x)\right|\right\} \\ &\le \sup_{x\in[0,1]}\left\{\left|g(x) - p(x)\right| + \left|\frac{\varepsilon}{2M}W(x)\right|\right\} \\ &\le \sup_{x\in[0,1]}\left\{\left|g(x) - p(x)\right|\right\} + \sup_{x\in[0,1]}\left\{\left|\frac{\varepsilon}{2M}W(x)\right|\right\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2M}M = \varepsilon.\end{aligned}$$

□

Since $C([0,1])$ is a complete metric space we see that by the *Baire Category Theorem* our set A must have an empty interior. Showing that when compared to the dense residual set of nowhere differentiable functions,

functions with even a single point of differentiability are not common. This can be strange to think about but does show us a limitation of human visualization. Our definitions for continuity are much weaker than the one for differentiability so it makes sense for the latter to be less common, the impression that they should almost come hand in hand stems from our familiarity with “nice” functions.

5 Notions of Size

In chapter 3 and 4 we introduced the idea that in complete metric spaces first category sets could be viewed as small. What is a “small” set? A set theorist might say a countable set, whilst we’ve seen that to a topologist it is nowhere dense and first category sets. Ask an analyst and you might expect an answer that a set of measure 0 encapsulates the idea of “smallness”. The aim of this chapter is to compare the two notions and show that they do not have to coincide and that in fact may be diametrically opposed.

Definition 5.1. A σ-ideal on a set X is a collection of subsets of X containing $\varnothing$ and closed under arbitrary subsets and countable unions.

Recall that Theorem 2.1 tells us that first category sets have these properties and so the class of first category sets is an example of a σ-ideal. We now present another example, the class of nullsets(often called sets of measure zero).

Definition 5.2. The length of any interval I is denoted by $|I|$. A set $A \subset \mathbb{R}$ is a nullset if for every $\varepsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \varepsilon$.

It’s trivially seen that singletons are nullsets and that any subset of a nullset is also a nullset. But what about a countable union of nullsets?

Proposition 5.3. *Any countable union of nullsets is also a nullset*

Proof. Suppose A_i is a nullset for $i \in \mathbb{N}$. Then for each i we have a sequence of intervals I_{ij} where $j \in \mathbb{N}$ such that $A_i \subset \bigcup_{j\in\mathbb{N}} I_{ij}$ and $\sum_{j=1}^{\infty} \left|I_{ij}\right| < \frac{\varepsilon}{2^i}$. The set of all intervals I_{ij} covers A and we also have that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left|I_{ij}\right| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$. □

To see why this falls apart when looking at uncountable unions consider

$$[-1,1] = \bigcup_{x\in[0,1]} \{x\}.$$

Since singletons are nullsets, the right hand side is an uncountable union of nullsets. But the interval on the left has length 2.

It should be clear that every countable set is of first category and a nullset since they can be represented as a countable union of singletons. What about uncountable sets?

One such example is the **Cantor set** C. The set is constructed iteratively by deleting the open middle third of $[0,1]$, then deleting the open middle thirds of each of the intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ and so forth. Let F_n be the union of the 2^n closed intervals of length $\frac{1}{3^n}$ that remain at the n^{th} step. We have that $C = \bigcap_{n\in\mathbb{N}} F_n$. The set C is closed since it is the intersection of closed sets. We also claim that C has an empty interior. Suppose it did not, then it would contain an open set of length $l > 0$. However F_n and therefore C contain intervals with a maximum length of $\frac{1}{3^n}$ which is less than l for sufficient n. We see then that C is nowhere dense. That C is a nullset follows from the simple fact that the total length of intervals in F_n is $(\frac{2}{3})^n$, which is less than ε for large enough n.

Theorem 5.4. *The cantor set is uncountable*

Proof. The set C is a closed subset of $\mathbb{R}$ and hence is also complete as a metric space. Recall that statement 3 of the Baire category theorem told us that a complete metric space could not be first category. We now see a way to complete the proof. If we can show that any arbitrary $x \in C$ is nowhere dense in C then we are

done since then if C were countable it would be a countable union of nowhere dense sets and thus of first category, a contradiction.
Choose some $x \in C$, there are points in C arbitrarily close to x by how we've constructed C. This shows $C \setminus \{x\}$ is dense in C. But {x} is closed, and so by Lemma A.2 we have that {x} is nowhere dense. So we cannot have that $C = \bigcup_{i\in\mathbb{N}} x_i$, hence C is uncountable. □

What do we actually mean by small here though? Loosely, a nullset is small in the sense that we can enclose it in a sequence of intervals with total length less than some arbitrary $\varepsilon > 0$. If you randomly chose a point in an interval so that the probability it is any subinterval I is proportional to $|I|$, the probability it lies in any nullset is zero. A nowhere dense set is small in the sense that it is punctured by holes everywhere, sets of first category being built from these will always have a dense set of gaps. However the Baire category theorem also tells us that in complete metric spaces first category sets will also have an empty interior. No interval of $\mathbb{R}$ can be represented as the union of such sets. It turns out that these two notions don't have to coincide, the following Theorem shows that neither class need to include the other.

Theorem 5.5. *$\mathbb{R}$ can be decomposed into two complementary sets A and A such that A is of first category and B is a nullset.*

Proof. (Oxtoby 1980, p. 5) Let $a_1, a_2, \dots$ be an enumeration of some countable dense subset of $\mathbb{R}$. Let I_{ij} be the open interval with center a_i and length $\frac{1}{2^{i+j}}$. Now for $j \in \mathbb{N}$ let $G_j = \bigcup_{i\in\mathbb{N}}$ and $B = \bigcap_{j\in\mathbb{N}}$. Note that for all $\varepsilon > 0$ we can choose a large enough $j \in \mathbb{N}$ such that $\frac{1}{2^j} < \varepsilon$. So since $B \subset \bigcup_{i\in\mathbb{N}} I_{ij}$ and also $\sum_{i\in\mathbb{N}} \left|I_{ij}\right| = \sum_{i\in\mathbb{N}} \frac{1}{2^{i+j}} = \frac{1}{2^j} < \varepsilon$ we see that B is a nullset. We now set $A = (\mathbb{R} \setminus B) = \bigcup_{j\in\mathbb{N}} (\mathbb{R} \setminus G_j)$. Notice that G_j is a dense open subset of $\mathbb{R}$ since it is the union of open intervals and contains every point of a dense subset. Hence, by Lemma A.3 we have that $(\mathbb{R} \setminus G_j)$ is nowhere dense. Thus A is a set of first category. □

This is a very striking result however it is not the first time we see it in this essay. Recall from Chapter 3 that the Lioville numbers formed a residual set in $\mathbb{R}$, we claim that it is also a nullset. Showing that L and $\mathbb{R} \setminus L$ decompose the real line in the manner specified in the previous theorem.

Theorem 5.6. *(Oxtoby 1980, p. 8) The set of Liouville numbers L is a nullset*

Proof. Remember that we defined L to be the following:

$$L = \Big(\mathbb{R} \setminus \mathbb{Q}\Big) \cap \bigcap_{n\in\mathbb{N}} U_n.$$

With U_n as:

$$U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \Big(-\frac{1}{q^n} + \frac{p}{q}, \frac{1}{q^n} + \frac{p}{q}\Big).$$

It will prove useful to also consider the following where $q \geq 2$ is a fixed integer,

$$U_{n,q} = \bigcup_{p=-\infty}^{\infty} \Big(-\frac{1}{q^n} + \frac{p}{q}, \frac{1}{q^n} + \frac{p}{q}\Big).$$

From the definition of L, one can see $L \subset U_n$ for every n and thus we have that for any positive integers m and n

$$L \cap (-m, m) \subset U_n \cap (-m, m) = \bigcup_{q=2}^{q=\infty} [U_{n,q} \cap (-m, m)] \subset U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} \Big(-\frac{1}{q^n} + \frac{p}{q}, \frac{1}{q^n} + \frac{p}{q}\Big).$$

We see that $L \cap (-m, m)$ is covered by a sequence of intervals. What is the sequence's length? For $n > 2$, we have that

$$\sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} \frac{2}{q^n} = \sum_{q=2}^{\infty} \frac{2}{q^n} \sum_{p=-mq}^{mq} 1 = \sum_{q=2}^{\infty} \Big(\frac{4m}{q^{n-1}} + \frac{2}{q^n}\Big) \leq \sum_{q=2}^{\infty} \Big(\frac{4m}{q^{n-1}} + \frac{q}{q^n}\Big) = (4m+1) \sum_{q=2}^{\infty} \frac{1}{q^{n-1}}$$
$$\leq (4m+1) \int_1^{\infty} \frac{1}{x^{n-1}}\, dx = (4m+1) \left[\frac{x^{2-n}}{2-n}\right]_1^{\infty} = \frac{4m+1}{n-2}.$$

So for any choice of m we can find a sufficiently large n such that $L \cap (-m, m)$ is covered by a sequence of intervals with total length less than ε. Hence L is a nullset.

□

Appendix

A Preliminaries

We recall the following notations and results from MA260 Norms, Metrics and Topologies:

- We denote the closure of a set A by $\overline{A}$
- We denote the interior of a set A by A°
- We denote the boundary of a set A by ∂A

Lemma A.1. *If $A \subset X$ then*

$$A^{\circ} = X \setminus \overline{(X \setminus A)} \quad \textit{and} \quad \overline{A} = X \setminus (X \setminus A)^{\circ}$$

Proof. (Robinson et al. 2023, p. 33) □

Lemma A.2. *A subset $A \subset X$ is nowhere dense in X if and only if $X \setminus \overline{A}$ is dense in X.*

Proof. Following the proof given in (Robinson et al. 2023, p. 34), by applying Lemma A.1 we have

$$\begin{aligned} A \text{ is nowhere dense} &\iff (\overline{A})^{\circ} = \varnothing \\ &\iff X \setminus \overline{(X \setminus \overline{A})} = \varnothing \\ &\iff \overline{(X \setminus \overline{A})} = X \\ &\iff (X \setminus \overline{A}) \text{ is dense in } X \end{aligned}$$

□

Lemma A.3. *A subset $A \subset X$ is nowhere dense in X if and only if it's complement contains a dense open set.*

Proof. By a slightly different application of Lemma A.1 we have

$$\begin{aligned} A \text{ is nowhere dense} &\iff (\overline{A})^{\circ} = \varnothing \\ &\iff X \setminus (\overline{A})^{\circ} = X \\ &\iff \overline{X \setminus \overline{A}} = X \\ &\iff \overline{(X \setminus A^{\circ})} = X \\ &\iff (X \setminus A)^{\circ} \text{ is dense in } X \\ &\iff (X \setminus A) \text{ contains a dense open set.} \end{aligned}$$

For the last equivalence note that for the forward direction the interior of a set is an open subset of the set itself. For the converse, if a set contains a dense open set U then it's interior must also contain U and hence is dense as well.

□

Lemma A.4. *A set is dense if and only if it's complement has empty interior*

Proof. Let A be a dense subset of X and once again apply Lemma A.1

$$\begin{aligned} A \text{ is dense } &\iff \overline{A} = X \\ &\iff X \setminus (X \setminus A)^{\circ} = X \\ &\iff (X \setminus A)^{\circ} = \varnothing. \end{aligned}$$

□

B Euler's number

Theorem B.1. *e is a transcendental number*

Proof. The first proof of this was given by Hermite (1873) who actually originally proved that e^γ was transcendental for all rational $\gamma \neq 0$. The proof shown here draw largely from the ones given in (Havil 2012, p. 191) and (Gelfond 1960, p. 42). Hermite's proof follows from an identity for e^x. Suppose $f(x)$ is any polynomial in x and define

$$F(x) = \sum_{k=0}^{\infty} f^{(k)}(x)$$

This is clearly a polynomial of the same degree as $f(x)$ since all the derivatives will be 0 past some large enough k.

One can see that $F(x) - F'(x) = f(x)$ so note that

$$\begin{aligned} \frac{d}{dx}(e^{-x}F(x)) &= e^{-x}F'(x) - e^{-x}F(x) \\ &= -e^{-x}(F(x) - F'(x)) = -e^{-x}f(x). \end{aligned}$$

which now yields

$$\int_0^x e^{-t} f(t)\, dt = [-e^{-t}F(t)]_0^x = F(0) - e^{-x}F(x) \tag{7}$$

For a contradiction we will assume that e is algebraic. So we have some polynomial of degree n with integer coefficients($a_0 \neq 0$) such that:

$$a_0 + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0$$

Now, take set $x = k = 0, 1, 2, \ldots, n$ in (7), multiply the equation by a_k and sum them together to reach

$$\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt = F(0) \sum_{k=0}^{n} a_k e^k - \sum_{k=0}^{n} a_k F(k)$$

However using our assumption we see that the first sum on the RHS is 0 and we can also isolate the first term in the second sum so we have that

$$a_0 F(0) + \sum_{k=1}^{n} a_k F(k) = -\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt \tag{8}$$

We are now in a position to choose f(t). We want to choose it so that the LHS of (8) is a non-zero integer, whilst making the RHS as small as possible which will give us our contradiction. Hermite's brilliantly clever polynomial is

$$f(t) = \frac{1}{(p-1)!} t^{p-1} g(t)^p, \quad g(t) = (t-1)(t-2)\ldots(t-n), \quad p > n + |a_0| \tag{9}$$

where p is a prime. The inequality for the size of p might seem to come out of nowhere however we'll shortly see why it is necessary.

We begin by showing that the LHS of (8) is a non-zero integer. Firstly lets look at $F(0)$. As $f(t)$ has a zero of multiplicity $p-1$ at $t = 0$ we see that $f^{(k)}(0) = 0$ for $k < p-1$, we can write the Taylor expansion(this is obviously a finite sum due to f being a polynomial) of f around 0 as

$$f(t) = \frac{1}{(p-1)!} f^{(p-1)}(0) t^{p-1} + \cdots + \frac{1}{k!} f^{(k)}(0) t^k + \ldots \tag{10}$$

When $k = p-1$ we have our first non zero value, which is $f^{(p-1)}(0) = (-1)^p \ldots (-n)^p = [(-1)^n n!]^p$. This is seen by comparing the coefficients of (9) and (10), see that the first term in (10) is the same as term given by the constant term of $g(t)$. $f^{(p-1)}(0)$ is clearly an integer but it is also important to note that it is not a muliple of p. This is due to minimum size we require p to be. since $p > n$, p will not occur in the prime factorisation of $n!$. What of the terms given by the the derivatives when $k \geq p$? Proceeding in the same manner as before and comparing the coefficients of each power gives us

$$f^{(k)}(0) = \frac{Ck!}{(p-1)!} \tag{11}$$

Where C is the coefficient for the (t^{k-p+1}) term in $g(t)^p$. C is obviously an integer and since $k \geq p$ we have that the entire thing is an integer. Thus $F(0)$ is an integer and since all terms in it's sum are divisible by p bar $f^{(k)}(0)$ we see it must also be non-zero.
We now look at the second term in the LHS, essentially repeating the method above but with Taylor expansions about $1, 2, \ldots, n$ instead. Since $f(t)$ has a zero of multiplicity p at $t = m$ where $1 \leq m \leq n$, $m \in \mathbb{Z}$ we have that $f^{(k)}(m) = 0$ for $0 \leq k < p$. For $k \geq p$ we play the same game as before and compare coefficients with (9), this gives us something very similar to (11)[5]. It follows that $F(m)$ is an integer and also a multiple of p for each $t = m$ where $1 \leq m \leq n$, $m \in \mathbb{Z}$. We can conclude by seeing that since $p > |a_0|$ we have that $a_0F(0)$ is an integer which isn't divisible by the prime p, the sum $\sum_1^n a_kF(k)$ is however an integer that is divisible by p. This combination finally gives us that the LHS of (8) is a non zero integer. That is, it must have an absolute value greater than or equal to 1 for some arbitrary p such that $p > n + |a_0|$.

We pass our attention to the RHS of (8), note that over the interval $[0, n]$ we have an upper bound for our large $f(t)$ can get

$$\left|f(t)\right| \leq \frac{n^{p-1}(n^p n^p \ldots n^p)}{(p-1)!} = \frac{n^{np+p-1}}{(p-1)!}$$

We can now estimate the size of the RHS as follows

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt\right| &\leq \sum_{k=0}^{n} |a_k|\, e^k \int_0^k e^{-t} \left|f(t)\right|\, dt \\
&\leq e^n \sum_{k=0}^{n} |a_k| \int_0^k e^{-t} \frac{n^{np+p-1}}{(p-1)!}\, dt \\
&= e^n \frac{n^{np+p-1}}{(p-1)!} \sum_{k=0}^{n} |a_k| \int_0^k e^{-t}\, dt \leq e^n \frac{n^{np+p}}{(p-1)!} \sum_{k=0}^{n} |a_k|
\end{aligned}$$

Since both the coefficients a_k and the degree n are fixed there exists a constant A such that we have

$$\left|\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\, dt\right| \leq A \frac{n^{np+p}}{(p-1)!}$$

We see that if we choose a large enough prime[6] we can make the RHS of (8) as small as we like, certainly less than 1. This results in a contradiction due to the LHS of(8) being a non-zero integer. Hence e is transcendental!

□

9 years later the transcendence of π was proven by Lindemann (1882) using a combination of the methods shown by Hermite together with the infamous equation $e^{i\pi} + 1 = 0$ [7].

[5]The specific details are omitted here to conserve space, see (Gelfond 1960, p. 43) for the precise values of the higher order derivatives

[6]This of course uses the fact that there are infinitely many primes, first proven by Euclid

[7]A nice proof of this is given in (Havil 2012, p. 194)

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